Hypothesis Tests

Cambridge International AS & A Level - Mathematics 9709
Paper 6: Probability & Statistics 2 - Exam-style question bank

Original Steps to Conquer practice questions in the style of recent 9709 Paper 6 examinations - not past Cambridge questions. Marks are shown in brackets [ ]. Give non-exact answers to 3 significant figures (1 d.p. for angles) and show full working.

Questions

1 A coin is claimed to show heads with probability 0.2. In 20 tosses, heads is obtained 8 times. Test, at the 5 percent significance level, whether the probability of heads is greater than 0.2. [7]

2 Complaints arrive at a shop at a mean rate of 2 per day. On one day 6 complaints are received. Test at the 5 percent level whether the mean number of complaints per day has increased. [7]

3 A normal population has known standard deviation 10. A sample of 25 observations has mean 52. Test at the 5 percent significance level whether the population mean is greater than 50. [7]

4 A normal population has known standard deviation 12. A sample of 36 observations has mean 98. Test at the 5 percent significance level whether the population mean is different from 100. [7]

5 A manufacturer claims that 45 percent of customers choose a particular product. In a sample of 120 customers, 66 choose it. Use a normal approximation to test at the 5 percent level whether the true proportion is greater than 0.45. [8]

6 For a test of $H_0 : p = 0.2$ against $H_1 : p > 0.2$, using $X \sim B(20, p)$, find the critical region at the 5 percent level. [6]

7 For the test in Question 6, calculate the probability of a Type II error when the true value of p is 0.35. [5]

8 A lower-tailed test of $H_0 : \lambda = 5$ against $H_1 : \lambda < 5$ is based on a single observation $X \sim \mathrm{Po}(\lambda)$. The critical region is $X \le 1$. Find the probability of a Type I error and the probability of a Type II error when $\lambda = 2.5$. [6]

9 A test of $H_0 : \mu = 100$ against $H_1 : \mu > 100$ is based on a sample of size 36 from a normal population with known standard deviation 12. Find the critical value of $\overline{X}$ for a 5 percent test. If the true mean is 106, find the probability of a Type II error. [8]

10 In the context of a hypothesis test about the mean fill volume of bottles, explain what is meant by a Type I error and a Type II error. [4]

Worked solutions

Full methods with final answers. Other valid methods may also earn full marks.

1 $H_0 : p = 0.2$ and $H_1 : p > 0.2$.

Under H_0, $X \sim B(20, 0.2)$.

The p-value is $P(X \geq 8)$.

$P(X \geq 8) = 0.0321$.

Since $0.0321 < 0.05$, reject H_0.

There is evidence at the 5 percent level that the probability of heads is greater than 0.2.

2 $H_0 : \lambda = 2$ and $H_1 : \lambda > 2$.

Under H_0, $X \sim \mathrm{Po}(2)$.

The p-value is $P(X \geq 6)$.

$P(X \geq 6) = 1 - P(X \leq 5) = 0.0166$.

Since $0.0166 < 0.05$, reject H_0.

There is evidence that the mean number of complaints per day has increased.

3 $H_0 : \mu = 50$ and $H_1 : \mu > 50$.

The test statistic is $Z = \dfrac{52 - 50}{10/\sqrt{25}} = 1.00$.

The p-value is $P(Z > 1.00) = 0.159$.

Since $0.159 > 0.05$, do not reject H_0.

There is insufficient evidence that the population mean is greater than 50.

4 $H_0 : \mu = 100$ and $H_1 : \mu \neq 100$.

$Z = \dfrac{98 - 100}{12/\sqrt{36}} = -1.00$.

For a two-tailed test, the p-value is $2P(Z < -1.00) = 0.317$.

Since $0.317 > 0.05$, do not reject H_0.

There is insufficient evidence that the population mean is different from 100.

5 $H_0 : p = 0.45$ and $H_1 : p > 0.45$.

Under H_0, $X \sim B(120, 0.45)$, with mean 54 and variance $120(0.45)(0.55) = 29.7$.

Using continuity correction, $P(X \geq 66) \approx P(Y > 65.5)$ where $Y \sim N(54, 29.7)$.

$z = \dfrac{65.5 - 54}{\sqrt{29.7}} = 2.11$.

The p-value is approximately 0.0174.

Since $0.0174 < 0.05$, reject H_0.

There is evidence that the true proportion is greater than 0.45.

6 Under H_0, $X \sim B(20, 0.2)$.

$P(X \geq 7) = 0.0867$, which is greater than 0.05.

$P(X \geq 8) = 0.0321$, which is less than 0.05.

The critical region is $X \geq 8$.

The actual significance level is 0.0321.

7 The critical region is $X \geq 8$.

A Type II error occurs when H_0 is not rejected although $p = 0.35$.

So the probability is $P(X \leq 7)$ for $X \sim B(20, 0.35)$.

$P(X \leq 7) = 0.601$ to 3 significant figures.

8 A Type I error occurs when H_0 is rejected while $\lambda = 5$.

$P(X \leq 1)$ for $X \sim \mathrm{Po}(5)$ is $e^{-5}(1+5) = 0.0404$.

A Type II error when $\lambda = 2.5$ occurs when H_0 is not rejected.

This is $P(X \geq 2)$ for $X \sim \mathrm{Po}(2.5)$.

$P(X \geq 2) = 1 - P(X \leq 1) = 0.713$ to 3 significant figures.

9 For a 5 percent upper-tailed test, use $z = 1.645$.

Reject H_0 when $\dfrac{\overline{X} - 100}{12/\sqrt{36}} > 1.645$.

Thus the critical value is $\overline{X} > 100 + 1.645(2) = 103.29$.

If $\mu = 106$, then $\overline{X} \sim N(106, 2^2)$.

A Type II error occurs when $\overline{X} \leq 103.29$.

$P(\overline{X} \leq 103.29) = P\left(Z \leq \dfrac{103.29 - 106}{2}\right) = 0.0877$.

10 A Type I error is rejecting the null hypothesis when it is actually true.

In context, this could mean concluding that the mean fill volume has changed when it has not changed.

A Type II error is failing to reject the null hypothesis when it is actually false.

In context, this could mean not detecting a real change in the mean fill volume.