Linear Combinations of Random Variables

Cambridge International AS & A Level - Mathematics 9709
Paper 6: Probability & Statistics 2 - Exam-style question bank

Original Steps to Conquer practice questions in the style of recent 9709 Paper 6 examinations - not past Cambridge questions. Marks are shown in brackets [ ]. Give non-exact answers to 3 significant figures (1 d.p. for angles) and show full working.

Questions

1 A random variable X has $E(X) = 10$ and $\mathrm{Var}(X) = 4$. Find $E(3X + 2)$ and $\mathrm{Var}(3X + 2)$. [4]

2 Independent random variables X and Y have $E(X) = 5$, $\mathrm{Var}(X) = 2$, $E(Y) = 8$ and $\mathrm{Var}(Y) = 3$. Find $E(2X - 3Y + 5)$ and $\mathrm{Var}(2X - 3Y + 5)$. [5]

3 The random variable X has distribution $N(20, 3^2)$. Find the distribution of $2X - 5$. [4]

4 Independent random variables X and Y have distributions $N(12, 4^2)$ and $N(7, 3^2)$ respectively. Find $P(X + Y > 20)$. [6]

5 Independent random variables X and Y have distributions $N(50, 4^2)$ and $N(32, 3^2)$ respectively. Find $P(X - Y > 5)$. [6]

6 The random variable X has distribution $N(100, 15^2)$. A random sample of 9 observations is taken and $\overline{X}$ is the sample mean. Find the distribution of $\overline{X}$ and find $P(\overline{X} > 102)$. [6]

7 Independent random variables X and Y have distributions $\mathrm{Po}(1.2)$ and $\mathrm{Po}(2.3)$ respectively. Find the distribution of $X + Y$ and calculate $P(X + Y \leq 3)$. [6]

8 Independent random variables X and Y have $E(X) = 4$, $\mathrm{Var}(X) = 1.5$, $E(Y) = 6$ and $\mathrm{Var}(Y) = 2.5$. Find the mean and variance of $5X + 2Y - 7$. [5]

9 A full carton contains 12 independent packets. Each packet has mass normally distributed with mean 250 g and standard deviation 8 g. Find the distribution of the total mass of the 12 packets. [4]

10 Independent random variables X and Y have distributions $N(10, 2^2)$ and $N(10, 6^2)$ respectively. Find $P(2X + Y < 32)$. [6]

Worked solutions

Full methods with final answers. Other valid methods may also earn full marks.

1 $E(3X+2) = 3E(X) + 2 = 3(10) + 2 = 32$.

$\mathrm{Var}(3X+2) = 3^2\,\mathrm{Var}(X) = 9(4) = 36$.

2 $E(2X - 3Y + 5) = 2(5) - 3(8) + 5 = -9$.

Since X and Y are independent, variances add for linear combinations.

$\mathrm{Var}(2X - 3Y + 5) = 2^2(2) + (-3)^2(3) = 8 + 27 = 35$.

3 If X is normal, then $2X - 5$ is also normal.

The mean is $2(20) - 5 = 35$.

The variance is $2^2(3^2) = 36$.

Thus $2X - 5 \sim N(35, 36)$.

4 $X + Y \sim N(12 + 7, 4^2 + 3^2) = N(19, 25)$.

$$P(X + Y > 20) = P\left(Z > \frac{20 - 19}{5}\right).$$

$= P(Z > 0.20) = 0.421$ to 3 significant figures.

5 $X - Y \sim N(50 - 32, 4^2 + 3^2) = N(18, 25)$.

$$P(X - Y > 5) = P\left(Z > \frac{5 - 18}{5}\right).$$

$= P(Z > -2.60) = 0.995$ to 3 significant figures.

6 $$\overline{X} \sim N\left(100, \frac{15^2}{9}\right) = N(100, 25).$$

$$P(\overline{X} > 102) = P\left(Z > \frac{102 - 100}{5}\right).$$

$= P(Z > 0.40) = 0.345$ to 3 significant figures.

7 The sum of independent Poisson random variables is Poisson.

$X + Y \sim \mathrm{Po}(1.2 + 2.3) = \mathrm{Po}(3.5)$.

$$P(X + Y \le 3) = e^{-3.5}\left(1 + 3.5 + \frac{3.5^2}{2!} + \frac{3.5^3}{3!}\right).$$

$P(X + Y \le 3) = 0.537$ to 3 significant figures.

8 $E(5X + 2Y - 7) = 5(4) + 2(6) - 7 = 25$.

$\mathrm{Var}(5X + 2Y - 7) = 5^2(1.5) + 2^2(2.5)$.

$= 37.5 + 10 = 47.5$.

9 Let T be the total mass.

Since the packet masses are independent normal random variables, T is normal.

$E(T) = 12(250) = 3000$.

$\mathrm{Var}(T) = 12(8^2) = 768$.

So $T \sim N(3000, 768)$.

10 $2X+Y$ is normally distributed.

Its mean is $2(10)+10=30$.

Its variance is $2^2(2^2)+6^2=16+36=52$.

$P(2X+Y<32)=P\left(Z<\frac{32-30}{\sqrt{52}}\right)$.

$=P(Z<0.277)=0.609$ to 3 significant figures.