Complex Numbers

Cambridge International AS & A Level - Mathematics 9709
Paper 3: Pure Mathematics 3 - Exam-style question bank

Original Steps to Conquer practice questions in the style of recent 9709 Paper 3 examinations - not past Cambridge questions. Marks are shown in brackets [ ]. Give non-exact answers to 3 significant figures (1 d.p. for angles) and show full working.

Questions

1 Let $z_1 = 3 + 2i$ and $z_2 = 1 - 4i$. Find $z_1 z_2$ and $\frac{z_1}{z_2}$ in Cartesian form. [5]

2 Find the modulus and argument of $z = -2 + 2\sqrt{3}, i$. Give the argument in radians, where $0 \le \arg z < 2\pi$. [4]

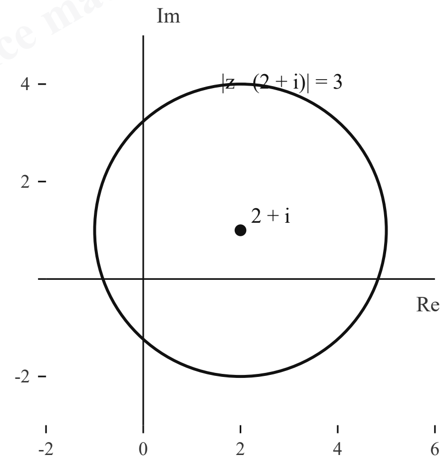
3 Solve the equation $z^2 - 4z + 13 = 0$. [4]

4 A cubic polynomial with real coefficients has roots $1 + 2i$ and -3. Find the polynomial in the form $z^3 + az^2 + bz + c$. [6]

5 Find the two square roots of $5 + 12i$, giving your answers in exact Cartesian form. [6]

6 Given $z_1 = 2(\cos 30^\circ + i \sin 30^\circ)$ and $z_2 = 3(\cos 80^\circ + i \sin 80^\circ)$, find $z_1 z_2$ in polar form. [3]

7 The locus $|z - (2 + i)| = 3$ is shown on the Argand diagram below. Describe the locus fully. [3]



Argand diagram for Question 7.

8 Find the Cartesian equation of the locus $|z - 1| = |z + 3i|$, where $z = x + iy$. [6]

9 Describe the region represented by $|z - 2| < |z + 2|$, where $z = x + iy$. [5]

10 The complex number z is transformed to $w = (1 + i)z$. Describe the geometrical effect of this transformation and find the image of $z = 2 - i$. [5]

Worked solutions

Full methods with final answers. Other valid methods may also earn full marks.

1 $z_1z_2 = (3+2i)(1-4i) = 3-12i+2i-8i^2 = 11-10i$.

$\dfrac{z_1}{z_2} = \dfrac{3+2i}{1-4i}\cdot\dfrac{1+4i}{1+4i}$.

$= \dfrac{3+12i+2i+8i^2}{17} = \dfrac{-5+14i}{17}$.

So $\dfrac{z_1}{z_2} = -\dfrac{5}{17}+\dfrac{14}{17}i$.

2 $|z| = \sqrt{(-2)^2+(2\sqrt{3})^2} = 4$.

The point is in the second quadrant and $\tan\theta = \sqrt{3}$.

Hence $\arg z = \dfrac{2\pi}{3}$.

3 $z = \dfrac{4 \pm \sqrt{16-52}}{2}$.

$z = \dfrac{4 \pm \sqrt{-36}}{2}$.

$z = 2 \pm 3i$.

4 Since the coefficients are real, $1-2i$ is also a root.

The quadratic factor from the complex roots is $(z-(1+2i))(z-(1-2i))$.

$= (z-1)^2+4 = z^2-2z+5$.

Including the root -3, the polynomial is $(z+3)(z^2-2z+5)$.

$= z^3+z^2-z+15$.

5 Let a square root be $a+bi$.

$(a+bi)^2 = a^2-b^2+2abi = 5+12i$.

So $a^2-b^2=5$ and $2ab=12$.

Also $a^2+b^2 = |5+12i| = 13$.

Adding gives $2a^2=18$, so $a=\pm 3$.

Then $b=\pm 2$ with the same sign as a.

The square roots are $3+2i$ and $-3-2i$.

6 When complex numbers in polar form are multiplied, moduli multiply and arguments add.

$|z_1z_2| = 2\cdot 3 = 6$.

$\arg(z_1z_2) = 30^\circ+80^\circ = 110^\circ$.

$z_1z_2 = 6(\cos 110^\circ + i\sin 110^\circ)$.

7 $|z-(2+i)| = 3$ means the distance from z to the point $2+i$ is 3.

The locus is a circle with centre $2+i$ and radius 3.

8 $|z-1|^2 = (x-1)^2+y^2$.

$|z+3i|^2 = x^2+(y+3)^2$.

Equating gives $(x-1)^2+y^2=x^2+(y+3)^2$.

$x^2-2x+1+y^2=x^2+y^2+6y+9$.

$-2x+1=6y+9$, so $x+3y+4=0$.

9 $|z-2|<|z+2|$ means the point z is closer to 2 than to -2 on the real axis.

Algebraically, $(x-2)^2+y^2<(x+2)^2+y^2$.

$x^2-4x+4<x^2+4x+4$.

$-8x<0$, so $x>0$.

The region is the half-plane to the right of the imaginary axis, excluding the axis.

10 $1+i=\sqrt{2}(\cos 45^\circ+i\sin 45^\circ)$.

Multiplication by $1+i$ is an enlargement with scale factor $\sqrt{2}$ and a rotation of 45° anticlockwise about the origin.

For $z=2-i$, $w=(1+i)(2-i)$.

$w=2-i+2i-i^2=3+i$.