Differential Equations

Cambridge International AS & A Level - Mathematics 9709
Paper 3: Pure Mathematics 3 - Exam-style question bank

Original Steps to Conquer practice questions in the style of recent 9709 Paper 3 examinations - not past Cambridge questions. Marks are shown in brackets [ ]. Give non-exact answers to 3 significant figures (1 d.p. for angles) and show full working.

Questions

1 Solve the differential equation $\frac{dy}{dx} = 3x^2y$, given that $y = 2$ when $x = 0$. [5]

2 A population P satisfies $\frac{dP}{dt} = kP$. Initially $P = 50$, and after 4 hours $P = 80$. Find P in terms of t, and find the time when $P = 120$. [7]

3 A cooling object has temperature T satisfying $\frac{dT}{dt} = -0.2(T - 18)$. Given that $T = 90$ when $t = 0$, find T in terms of t and the time when $T = 30$. [6]

4 Solve $\frac{dy}{dx} = \frac{x+1}{y}$, given that $y = 3$ when $x = 0$. [5]

5 A tank contains V litres of water at time t minutes. The rate of decrease of V is proportional to $\sqrt{V}$. Initially $V = 100$, and after 9 minutes $V = 25$. Find when the tank is empty. [7]

6 A quantity N increases at a rate proportional to $200 - N$. Initially $N = 50$, and $N = 100$ after 5 minutes. Form and solve the differential equation for N. [7]

7 Solve $\frac{dy}{dx} = xe^{-y}$, given that $y = 0$ when $x = 0$. [5]

8 Solve $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$, given that $y = 1$ when $x = 0$. [6]

9 A mass M satisfies $\frac{dM}{dt} = -kM^2$. Initially $M = 10$, and after 5 seconds $M = 8$. Find M in terms of t, and find when $M = 5$. [7]

10 Solve $\frac{dy}{dx} = \frac{2x+1}{y+2}$, given that $y = 1$ when $x = 0$. [5]

Worked solutions

Full methods with final answers. Other valid methods may also earn full marks.

1 $\frac{1}{y}\,dy = 3x^2\,dx$.

$\ln y = x^3 + C$.

Using $y = 2$ when $x = 0$ gives $C = \ln 2$.

Therefore $y = 2e^{x^3}$.

2 Solving gives $P = Ae^{kt}$.

Since $P = 50$ at $t = 0$, $A = 50$.

$80 = 50e^{4k}$, so $e^{4k} = \frac{8}{5}$ and $k = \frac{1}{4}\ln(\frac{8}{5})$.

$P = 50e^{(t/4)\ln(8/5)}$.

For $P = 120$, $e^{kt} = \frac{12}{5}$.

$t = \dfrac{\ln(12/5)}{k} = \dfrac{4\ln(12/5)}{\ln(8/5)} = 7.45$ hours to 3 significant figures.

3 $\dfrac{dT}{T-18} = -0.2\,dt$.

$\ln(T-18) = -0.2t + C$.

$T - 18 = Ae^{-0.2t}$.

Using $T = 90$ when $t = 0$ gives $A = 72$.

$T = 18 + 72e^{-0.2t}$.

For $T = 30$, $12 = 72e^{-0.2t}$, so $t = 5\ln 6 = 8.96$.

4 $y\,dy = (x+1)\,dx$.

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C$.

Using $y = 3$ when $x = 0$ gives $C = \frac{9}{2}$.

$y^2 = x^2 + 2x + 9$.

Since $y = 3$ initially, $y = \sqrt{x^2 + 2x + 9}$ for the branch through the initial point.

5 $\dfrac{dV}{dt} = -k\sqrt{V}$.

$\dfrac{dV}{\sqrt{V}} = -k\,dt$.

$2\sqrt{V} = -kt + C$.

At $t = 0$, $V = 100$, so $C = 20$.

At $t = 9$, $V = 25$, so $10 = -9k + 20$, giving $k = \frac{10}{9}$.

When the tank is empty, $0 = -\frac{10}{9}t + 20$.

$t = 18$ minutes.

6 $\dfrac{dN}{dt} = k(200 - N)$.

$\dfrac{dN}{200 - N} = k\,dt$.

$-\ln(200-N)=kt+C$, so $200-N=Ae^{-kt}$.

Using $N=50$ at $t=0$ gives $A=150$.

Using $N=100$ at $t=5$ gives $100=150e^{-5k}$.

$e^{-5k}=\frac{2}{3}$, so $k=\frac{1}{5}\ln(\frac{3}{2})$.

$N=200-150e^{-t\ln(3/2)/5}$.

7 $e^y\,dy=x\,dx$.

$e^y=\frac{1}{2}x^2+C$.

Using $y=0$ when $x=0$ gives $C=1$.

$e^y=1+\frac{1}{2}x^2$.

Hence $y=\ln(1+\frac{1}{2}x^2)$.

8 $\dfrac{dy}{1+y^2}=\dfrac{dx}{1+x^2}$.

$\tan^{-1}y=\tan^{-1}x+C$.

Using $y=1$ when $x=0$ gives $\frac{\pi}{4}=C$.

$\tan^{-1}y=\tan^{-1}x+\frac{\pi}{4}$.

Taking tangent, $y=\dfrac{x+1}{1-x}$, for the interval where the expression is defined.

9 $\int M^{-2}\,dM=\int -k\,dt$.

$-\dfrac{1}{M}=-kt+C$, so $\dfrac{1}{M}=kt+C_1$.

Using $M=10$ at $t=0$ gives $C_1=\frac{1}{10}$.

Using $M=8$ at $t=5$ gives $\frac{1}{8}=5k+\frac{1}{10}$.

$k=\frac{1}{200}$.

$\dfrac{1}{M}=\dfrac{t}{200}+\dfrac{1}{10}=\dfrac{t+20}{200}$, so $M=\dfrac{200}{t+20}$.

When $M=5$, $5=\dfrac{200}{t+20}$, so $t=20$ seconds.

10 $(y+2)\,dy=(2x+1)\,dx$.

$\frac{1}{2}y^2+2y=x^2+x+C$.

Using $y=1$ when $x=0$ gives $C=\frac{5}{2}$.

$y^2+4y=2x^2+2x+5$.

$(y+2)^2=2x^2+2x+9$, so $y=-2+\sqrt{2x^2+2x+9}$ for the required branch.