Vectors

Cambridge International AS & A Level - Mathematics 9709
Paper 3: Pure Mathematics 3 - Exam-style question bank

Original Steps to Conquer practice questions in the style of recent 9709 Paper 3 examinations - not past Cambridge questions. Marks are shown in brackets [ ]. Give non-exact answers to 3 significant figures (1 d.p. for angles) and show full working.

Questions

1 The points A and B have coordinates $(1, 2, -1)$ and $(4, -2, 3)$ respectively. Find $\overrightarrow{AB}$, $|\overrightarrow{AB}|$, and a unit vector in the direction of $\overrightarrow{AB}$. [5]

2 Find the vector equation of the line passing through $P(2, -1, 4)$ and $Q(5, 3, 2)$. [4]

3 The lines l_1 and l_2 are given by $\mathbf{r} = (1, 0, 2) + s(2, 1, -1)$ and $\mathbf{r} = (3, 1, 1) + t(1, 0, 2)$. Show that the lines intersect and find the angle between them. [7]

4 The lines l_1 and l_2 are defined by $\mathbf{r} = (1, 2, 0) + s(1, -1, 2)$ and $\mathbf{r} = (3, 0, 1) + t(2, 1, -1)$. Show that the lines are skew. [6]

5 Find the angle between the vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. [5]

6 The point $P(1, 2, 3)$ is joined perpendicularly to the line $\mathbf{r} = (2, 0, 1) + t(1, 1, 2)$. Find the coordinates of the foot of the perpendicular. [7]

7 In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P lies on AB such that $AP : PB = 2 : 1$. Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

8 The line l has equation $\mathbf{r} = (1, 2, -1) + t(2, -1, 2)$. Find the two points on l which are distance 6 from $(1, 2, -1)$. [5]

9 Find the value of k for which the vectors $(k, 2, 1)$ and $(3, k, -2)$ are perpendicular. [4]

10 The points $A(1, 0, 2)$, $B(3, 1, 4)$ and $C(2, -1, 5)$ form a triangle. Find $\cos \angle BAC$. [5]

Worked solutions

Full methods with final answers. Other valid methods may also earn full marks.

1 $\overrightarrow{AB} = (4-1, -2-2, 3-(-1)) = (3, -4, 4)$.

$|\overrightarrow{AB}| = \sqrt{3^2 + (-4)^2 + 4^2} = \sqrt{41}$.

A unit vector in the direction of $\overrightarrow{AB}$ is $\frac{1}{\sqrt{41}}(3, -4, 4)$.

2 A direction vector is $\overrightarrow{PQ} = (3, 4, -2)$.

The line may be written as $\mathbf{r} = (2, -1, 4) + t(3, 4, -2)$.

3 For $s = 1$ on l_1, $\mathbf{r} = (3, 1, 1)$, which is the point on l_2 when $t = 0$.

So the lines intersect at $(3, 1, 1)$.

Direction vectors are $\mathbf{a} = (2, 1, -1)$ and $\mathbf{b} = (1, 0, 2)$.

$\mathbf{a} \cdot \mathbf{b} = 2 + 0 - 2 = 0$.

Therefore the angle between the lines is 90°.

4 The direction vectors $(1, -1, 2)$ and $(2, 1, -1)$ are not scalar multiples, so the lines are not parallel.

For an intersection, $1 + s = 3 + 2t$, $2 - s = t$, and $2s = 1 - t$.

From $2 - s = t$, substitute into $1 + s = 3 + 2t$ to get $1 + s = 3 + 2(2 - s)$.

Thus $3s = 6$, so $s = 2$ and $t = 0$.

But the third equation gives $2s = 4$ and $1 - t = 1$, which is inconsistent.

The lines are not parallel and do not intersect, so they are skew.

5 $\mathbf{a} \cdot \mathbf{b} = 2 - 3 - 4 = -5$.

$|\mathbf{a}| = 3$ and $|\mathbf{b}| = \sqrt{14}$.

$\cos\theta = \dfrac{-5}{3\sqrt{14}}$.

$\theta = 116.5^\circ$ to 1 decimal place.

6 Let $A = (2, 0, 1)$ and $\mathbf{d} = (1, 1, 2)$.

The foot has position vector $A + t\mathbf{d}$.

For perpendicularity, $(P - A - t\mathbf{d}) \cdot \mathbf{d} = 0$.

$P - A = (-1, 2, 2)$ and $(P - A) \cdot \mathbf{d} = -1 + 2 + 4 = 5$.

$\mathbf{d} \cdot \mathbf{d} = 6$, so $t = \frac{5}{6}$.

The foot is $(2, 0, 1) + \frac{5}{6}(1, 1, 2) = (\frac{17}{6}, \frac{5}{6}, \frac{8}{3})$.

7 P is two thirds of the way from A to B.

$\overrightarrow{OP} = \overrightarrow{OA} + \dfrac{2}{3}\overrightarrow{AB}$.

$= \mathbf{a} + \dfrac{2}{3}(\mathbf{b} - \mathbf{a})$.

$\overrightarrow{OP} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$.

8 The direction vector is $(2, -1, 2)$, whose magnitude is 3.

A point on the line is $(1, 2, -1) + t(2, -1, 2)$.

Its distance from $(1, 2, -1)$ is $3|t|$.

Set $3|t| = 6$, so $t = 2$ or $t = -2$.

The points are $(5, 0, 3)$ and $(-3, 4, -5)$.

9 Perpendicular vectors have scalar product zero.

$(k, 2, 1) \cdot (3, k, -2) = 3k + 2k - 2$.

$5k - 2 = 0$, so $k = \frac{2}{5}$.

10 $\overrightarrow{AB} = (2, 1, 2)$ and $\overrightarrow{AC} = (1, -1, 3)$.

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 2 - 1 + 6 = 7$.

$|\overrightarrow{AB}| = 3$ and $|\overrightarrow{AC}| = \sqrt{11}$.

$\cos \angle BAC = \frac{7}{3\sqrt{11}}$.