Numerical Solution of Equations

Cambridge International AS & A Level - Mathematics 9709
Paper 3: Pure Mathematics 3 - Exam-style question bank

Original Steps to Conquer practice questions in the style of recent 9709 Paper 3 examinations - not past Cambridge questions. Marks are shown in brackets [ ]. Give non-exact answers to 3 significant figures (1 d.p. for angles) and show full working.

Questions

1 Show that the equation $x^3 - x - 1 = 0$ has a root between 1 and 2. [3]

2 The equation $x^3 - x - 1 = 0$ is rearranged as $x = \sqrt[3]{x+1}$. Starting with $x_0 = 1$, use the iteration $x_{n+1} = \sqrt[3]{x_n + 1}$ to find x_4. Give your answer to 4 decimal places. [4]

3 Use the values of $f(x) = x^3 - x - 1$ at $x = 1.32$ and $x = 1.33$ to show that the root is 1.3 to 1 decimal place. [4]

4 The equation $x + \ln x = 3$ has a root α. Show that $2 < \alpha < 3$, and then use $x_{n+1} = 3 - \ln x_n$ with $x_0 = 2$ to find x_4. [5]

5 The equation $x = e^{-x}$ has a root. Starting with $x_0 = 0.5$, use the iteration $x_{n+1} = e^{-x_n}$ to find x_5. [4]

6 A root of $x^3 + 2x - 5 = 0$ is to be found using $x_{n+1} = \sqrt[3]{5 - 2x_n}$. Starting with $x_0 = 1.2$, calculate x_5. [4]

7 Explain briefly why the iteration $x_{n+1} = 3 - \ln x_n$ cannot be used if $x_n \le 0$. [2]

8 The sequence defined by $x_{n+1} = \sqrt{6 - x_n}$ converges to a positive limit L. Find the exact value of L. [4]

9 Starting with $x_0 = 1$, use $x_{n+1} = \sqrt{6 - x_n}$ to calculate x_5. Hence comment on the convergence towards the limit found in Question 8. [5]

10 A student uses an iterative formula to obtain 1.3281, 1.3283, 1.3283 as three successive approximations to a root. State the root to 3 decimal places and explain why this degree of accuracy is justified. [3]

Worked solutions

Full methods with final answers. Other valid methods may also earn full marks.

1 Let $f(x) = x^3 - x - 1$.

$f(1) = 1 - 1 - 1 = -1$ and $f(2) = 8 - 2 - 1 = 5$.

Since there is a change of sign, there is a root between 1 and 2.

2 $x_1 = \sqrt[3]{2} = 1.2599$.

$x_2 = \sqrt[3]{2.2599} = 1.3123$.

$x_3 = \sqrt[3]{2.3123} = 1.3224$.

$x_4 = \sqrt[3]{2.3224} = 1.3243$.

3 $f(1.32) = 1.32^3 - 1.32 - 1 = -0.020032$.

$f(1.33) = 1.33^3 - 1.33 - 1 = 0.022637$.

The root lies between 1.32 and 1.33.

All numbers in this interval round to 1.3 to 1 decimal place.

4 Let $f(x) = x + \ln x - 3$.

$f(2) = 2 + \ln 2 - 3 < 0$ and $f(3) = \ln 3 > 0$, so $2 < \alpha < 3$.

$x_1 = 3 - \ln 2 = 2.3069$.

$x_2 = 3 - \ln 2.3069 = 2.1641$.

$x_3 = 3 - \ln 2.1641 = 2.2280$.

$x_4 = 3 - \ln 2.2280 = 2.1989$.

5 $x_1 = e^{-0.5} = 0.6065$.

$x_2 = e^{-0.6065} = 0.5452$.

$x_3 = e^{-0.5452} = 0.5797$.

$x_4 = e^{-0.5797} = 0.5601$.

$x_5 = e^{-0.5601} = 0.5712$.

6 $x_1 = 1.3751$.

$x_2 = 1.3103$.

$x_3 = 1.3350$.

$x_4 = 1.3257$.

$x_5 = 1.3292$.

7 The expression $\ln x_n$ is defined only for $x_n > 0$.

If $x_n \le 0$, the next term x_{n+1} is not real, so the iteration cannot continue in the real number system.

8 At the limit, $L = \sqrt{6 - L}$.

Squaring gives $L^2 = 6 - L$.

$L^2 + L - 6 = 0$, so $(L + 3)(L - 2) = 0$.

Since L is positive, $L = 2$.

9 $x_1 = 2.2361$.

$x_2 = 1.9401$.

$x_3 = 2.0149$.

$x_4 = 1.9963$.

$x_5 = 2.0009$.

The values alternate around 2 and move closer to 2, so the iteration is converging to the positive limit.

10 The approximations agree to 3 decimal places as 1.328 .

Successive values are unchanged to 3 decimal places, so the root may be stated as 1.328 to 3 decimal places.