Numerical Solution of Equations

Cambridge International AS & A Level - Mathematics 9709
Paper 2: Pure Mathematics 2 - Exam-style question bank

Original Steps to Conquer practice questions in the style of recent 9709 Paper 2 examinations - not past Cambridge questions. Marks are shown in brackets [ ]. Give non-exact answers to 3 significant figures (1 d.p. for angles) and show full working.

Questions

1 Show that the equation $x^3 - 2x - 5 = 0$ has a root between 2 and 3. [3]

2 The equation $x^3 - 2x - 5 = 0$ is rearranged as $x = \sqrt[3]{2x+5}$. Starting with $x_0 = 2$, use $x_{n+1} = \sqrt[3]{2x_n + 5}$ to find x_4. [4]

3 Use the values of $f(x) = x^3 - 2x - 5$ at $x = 2.09$ and $x = 2.10$ to show that the root is 2.1 to 1 decimal place. [4]

4 The equation $x + \ln x = 4$ has a root α. Show that $2 < \alpha < 3$. [3]

5 The equation $x + \ln x = 4$ is rearranged as $x = 4 - \ln x$. Starting with $x_0 = 3$, use the iteration $x_{n+1} = 4 - \ln x_n$ to find x_4. [4]

6 The equation $e^{-x} = x$ has a root. Starting with $x_0 = 0.6$, use $x_{n+1} = e^{-x_n}$ to find x_5. [4]

7 A root of $x^2 + \ln x = 5$ is to be found. Show that it lies between 2 and 3. [3]

8 Explain why the iteration $x_{n+1} = \ln(5 - x_n^2)$ cannot be used if $x_n \geq \sqrt{5}$. [2]

9 A sequence is defined by $x_{n+1} = \sqrt{7 - x_n}$ and converges to a positive limit L. Find L. [4]

10 Successive approximations to a root are 2.0942, 2.0946 and 2.0945. State the root to 3 decimal places, giving a reason. [3]

Worked solutions

Full methods with final answers. Other valid methods may also earn full marks.

1 Let $f(x) = x^3 - 2x - 5$.

$f(2) = 8 - 4 - 5 = -1$ and $f(3) = 27 - 6 - 5 = 16$.

There is a change of sign, so a root lies between 2 and 3.

2 $x_1 = \sqrt[3]{9} = 2.0801$.

$x_2 = \sqrt[3]{9.1602} = 2.0923$.

$x_3 = \sqrt[3]{9.1846} = 2.0941$.

$x_4 = \sqrt[3]{9.1882} = 2.0944$.

3 $f(2.09) = 2.09^3 - 2(2.09) - 5 = -0.050$ approximately.

$f(2.10) = 2.10^3 - 2(2.10) - 5 = 0.061$.

The root lies between 2.09 and 2.10.

Therefore the root is 2.1 to 1 decimal place.

4 Let $f(x) = x + \ln x - 4$.

$f(2) = 2 + \ln 2 - 4 < 0$.

$f(3) = 3 + \ln 3 - 4 > 0$.

There is a change of sign, so $2 < \alpha < 3$.

5 $x_1 = 4 - \ln 3 = 2.9014$.

$x_2 = 4 - \ln 2.9014 = 2.9350$.

$x_3 = 4 - \ln 2.9350 = 2.9235$.

$x_4 = 4 - \ln 2.9235 = 2.9274$.

6 $x_1 = e^{-0.6} = 0.5488$.

$x_2 = e^{-0.5488} = 0.5776$.

$x_3 = e^{-0.5776} = 0.5612$.

$x_4 = e^{-0.5612} = 0.5704$.

$x_5 = e^{-0.5704} = 0.5653$.

7 Let $f(x) = x^2 + \ln x - 5$.

$f(2) = 4 + \ln 2 - 5 = -0.307$.

$f(3) = 9 + \ln 3 - 5 > 0$.

There is a change of sign, so a root lies between 2 and 3.

8 If $x_n \geq \sqrt{5}$, then $5 - x_n^2 \leq 0$.

The logarithm of a non-positive number is not real, so the iteration cannot continue in real numbers.

9 At the limit, $L = \sqrt{7 - L}$.

$L^2 = 7 - L$, so $L^2 + L - 7 = 0$.

$L = \dfrac{-1 \pm \sqrt{29}}{2}$.

The positive limit is $L = \dfrac{-1+\sqrt{29}}{2}$.

10 The approximations agree to 3 decimal places as 2.095 .

Since successive values are stable to 3 decimal places, the root is 2.095 to 3 decimal places.